8-3 Magnetic Materials

Reading Assignment: *pp. 244 - 260*

Recall in dielectrics, electric dipoles were created when and E-field was applied.

→ Therefore, we defined permittivity ε, electric flux density $\mathbf{D}(\bar{r})$, and a new set of electrostatic equations.

Q:

A:

8-3-1 Orbital and Spin Currents

HO: Magnetic Materials

HO: The Magnetic Dipole in a B-field

8-3-2 Magnetic Susceptibility and Magnetization Currents

HO: The Magnetization Vector

HO: Magnetization Currents

8-3-3 The Magnetic Field Intensity

HO: The Magnetic Field

Example: Magnetization Currents

8-3-4 The Physical Properties of Magnetic Materials

HO: Permanent Magnents

8-3-5 Field Equations in Magnetic Materials

HO: Field Equations in Magnetic Materials

8-3-6 Magnetic Field Boundary Conditions

HO: Magnetic Boundary Conditions

Magnetic Materials

Recall that **atoms and molecules**, having both positive (i.e., protons) and negative (i.e., electron) charged particles can form **electric dipoles**.

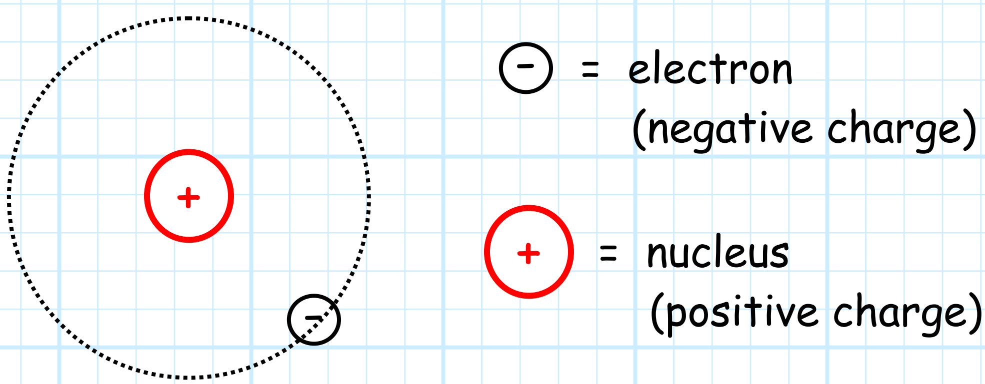
We find that atoms and molecules also can also form **magnetic dipoles**!

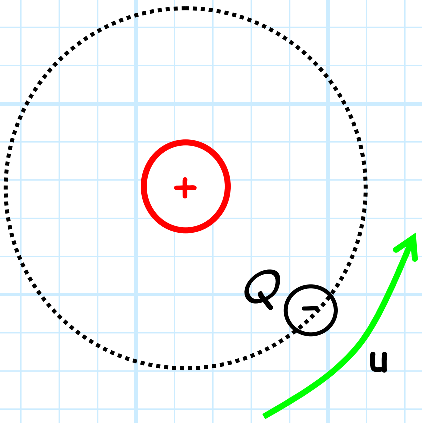
Q: *How??*

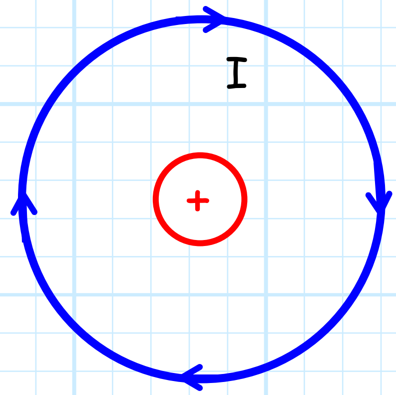
A: Recall a magnetic dipole is formed when current flows in a **small loop**. Current, of course, is **moving charge**, therefore charge moving around a small loop forms a magnetic dipole.

Molecules and atoms **often** exhibit electrons moving around in small loops!

Again, we use our **ridiculously** simple model of an atom:





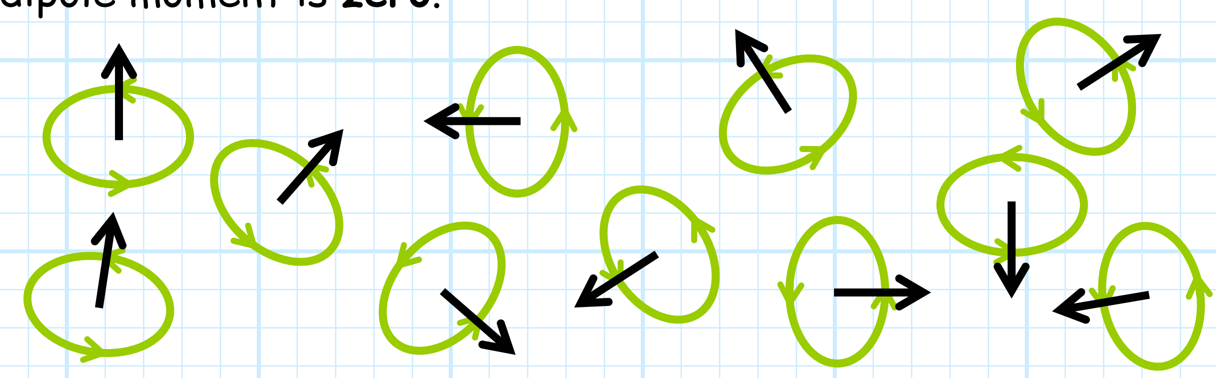


An electron with charge Q orbiting around a nucleus at velocity **u** forms a **small current loop**, where $I = Q|\mathbf{u}|$.

This forms a **magnetic dipole**!

This is a **very simple** atomic explanation of how magnetic dipoles are formed in material. In actuality, the physical mechanisms that lead to magnetic dipoles can be **far** more complex. For example, **electron spin** can also create a magnetic dipole moment.

Typically, the atoms/molecules of materials exhibit either **no** magnetic dipole moment (i.e., $\mathbf{m} = 0$), or the dipole moments of each atom/molecule are **randomly oriented**, such that the **net** dipole moment is **zero**.



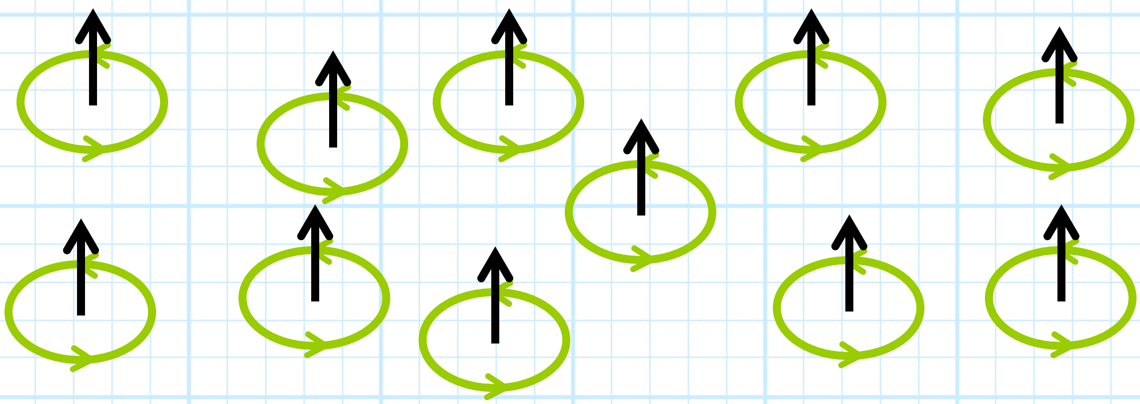
Therefore, if we have N randomly oriented magnetic dipoles $\mathbf{m}_n$, we find there average value will be zero:

$$\frac{1}{N}\sum_n \mathbf{m}_n = 0$$

Similarly, we find that the **total** magnetic flux density created by these magnetic dipoles is **also zero**:

$$\sum_n \mathbf{B}_n(\bar{r}) = 0$$

However, we find that sometimes the magnetic dipole moment of each atom/molecule is **not** randomly oriented, but in fact are **aligned**!



In this case, total magnetic flux density created by these dipoles is **non-zero**!

$$\sum_n \mathbf{B}_n(\bar{r}) \neq 0.$$

Q: ***Why** would these magnetic dipoles be aligned?*

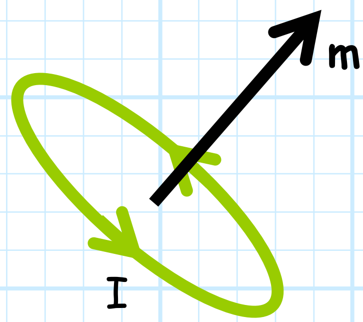
A: **Two** possible reasons:

1) the material is a **permanent magnet**.

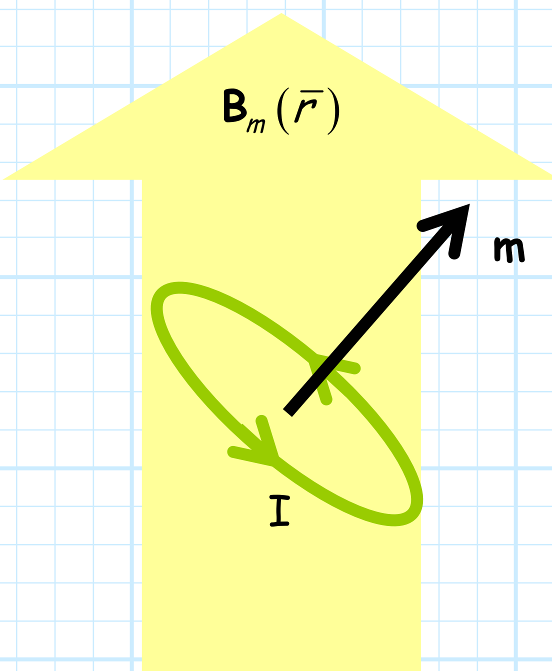
2) the material is immersed in some **magnetizing field** $\mathbf{B}_m(\bar{r})$.

The Magnetic Dipole in a B-field

Consider the case of an **arbitrarily aligned** magnetic dipole:



Say this dipole is **immersed** in some field $\mathbf{B}_m(\bar{r})$:



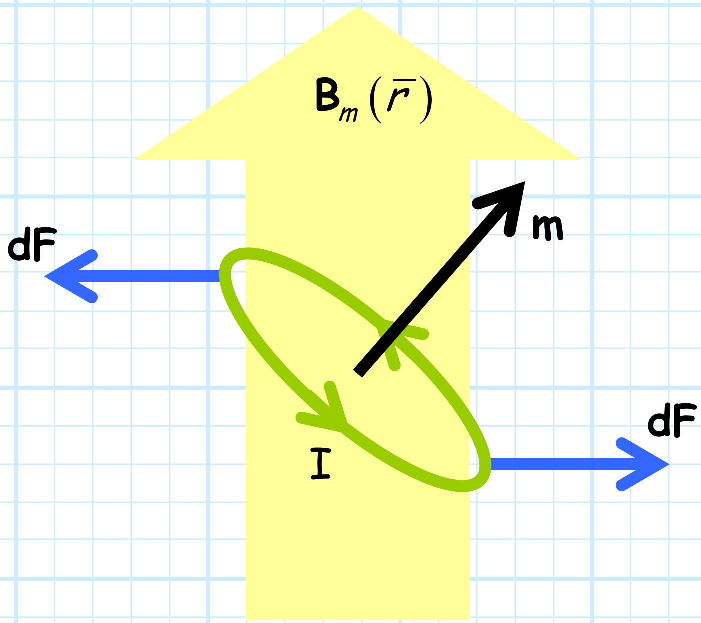
Q: *What happens to a **magnetic dipole** when exposed to a magnetic flux density* $\mathbf{B}_m(\bar{r})$*?*

A: Exactly what the **Lorentz Force equation** says will happen!

Recall that the force **dF** on some current element $I\,\overline{d\ell}$ is:

$$\mathbf{dF} = I\,\overline{d\ell} \times \mathbf{B}_m(\bar{r})$$

Note this force is therefore **perpendicular** to **both** $\mathbf{B}(\bar{r})$ and current I.

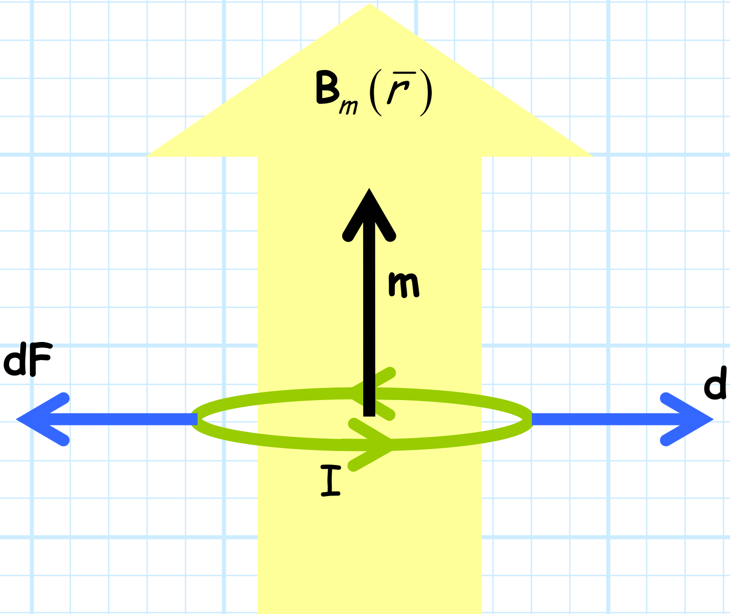


The total **resultant** force on a current loop is will be **zero**, so the dipole does **not** change position. I.E.:

$$\oint_C I\,\overline{d\ell} \times \mathbf{B}_m(\bar{r}) = 0$$

However, the forces on the current do apply a **torque** $\mathbf{T}_m$ to the current loop!

The current loop (i.e., magnetic dipole) will **rotate** until the dipole moment **m** is aligned with the magnetic flux density vector $\mathbf{B}_m(\bar{r})$.



For a **circular** current loop, it can be shown (pp. 234-235) that the torque applied is:

$$\mathbf{T}_m = \mathbf{m} \times \mathbf{B}(\bar{r}) \qquad [N \cdot m]$$

Note that once the magnetic dipole moment **m** is aligned with magnetic flux density $\mathbf{B}(\bar{r})$, the torque $\mathbf{T}_m$ is equal to **zero**—the magnetic dipole **stops rotating** and **remains aligned** with $\mathbf{B}(\bar{r})$.

The Magnetization Vector

Recall that we defined the **Polarization vector** of a dielectric material as the **electric dipole density**, i.e.:

$$\mathbf{P}(\bar{\mathrm{r}}) \doteq \lim_{\Delta v \to 0} \frac{\sum \mathbf{p}_n}{\Delta v} \qquad \left[\frac{\text{electric dipole moment}}{\text{unit volume}}\right]$$

Similarly, we can define a **Magnetization vector** $\mathbf{M}(\bar{r})$ of a material to be the density of **magnetic** dipole moments at location $\bar{r}$:

$$\mathbf{M}(\bar{\mathrm{r}}) \doteq \lim_{\Delta v \to 0} \frac{\sum \mathbf{m}_n}{\Delta v} \qquad \left[\frac{\text{magnetic dipole moment}}{\text{unit volume}} = \frac{A}{m}\right]$$

Note if the dipole moments of atoms/molecules within a material are **completely random**, the Magnetization vector will be **zero** (i.e., $\mathbf{M}(\bar{r}) = 0$).

However, if the dipoles are **aligned**, the Magnetization vector will be **non-zero** (i.e., $\mathbf{M}(\bar{r}) \neq 0$)

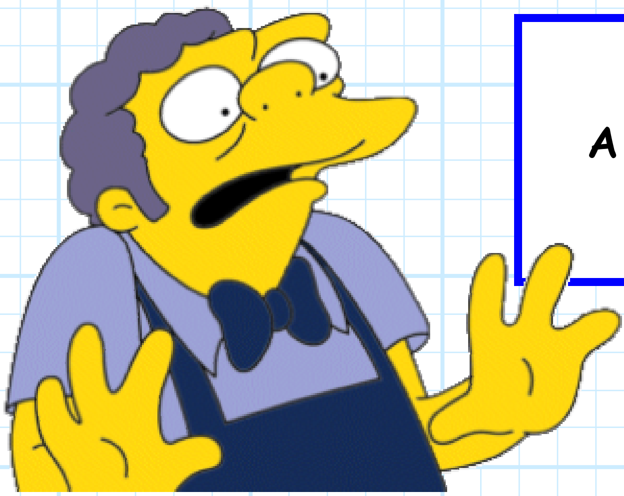
Recall a magnetic dipole will create a **magnetic vector potential** equal to:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0 \mathbf{m} \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$$

Since the magnetic dipole moment of some **small** (i.e., differential) volume dv of the material is:

$$\mathbf{m} = \mathbf{M}(\bar{r})\, dv$$

we find that the magnetic vector potential created by a **volume** V of material with magnetization vector $\mathbf{M}(\bar{r})$ is:



$$\mathbf{A}(\bar{r}) = \iiint_V \frac{\mu_0 \mathbf{M}(\bar{r}') \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}\, dv'$$

Q: *This is freaking me out!! I thought that **currents** $\mathbf{J}(\bar{r})$ were responsible for creating magnetic vector potential. In fact, I could have sworn that:*

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|}\, dv'$$

A: Relax, **both** expressions are correct!

Recall that we could attribute the electric field created by Polarization Vector $\mathbf{P}(\bar{r})$ to **polarization** (i.e., bound) **charges** $\rho_{vp}(\bar{r})$ and $\rho_{sp}(\bar{r})$, i.e., :

$$\rho_{vp}(\bar{\mathbf{r}}) = -\nabla \cdot \mathbf{P}(\bar{\mathbf{r}}) \qquad\qquad \rho_{sp}(\bar{\mathbf{r}}) = \mathbf{P}(\bar{\mathbf{r}}) \cdot \hat{a}_n$$

Similarly, we can **attribute** the magnetic vector potential (and therefore the magnetic flux density) created by Magnetization Vector $\mathbf{M}(\bar{r})$ to **Magnetization Currents** $\mathbf{J}_m(\bar{r})$ and $\mathbf{J}_{sm}(\bar{r})$.

Magnetization Currents

Recall that the magnetic vector potential $\mathbf{A}(\bar{r})$ created by **volume** current distribution $\mathbf{J}(\bar{r})$is:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \, dv'$$

while the magnetic vector potential created by a **surface** current $\mathbf{J}_s(\bar{r})$:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \, ds'$$

Therefore, if **both** volume and surface current densities are present we find that the **total** magnetic vector potential is:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \, dv' + \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \, ds'$$

Compare these expressions to the magnetic vector potential field produced by material with **Magnetization Vector $\mathbf{M}(\bar{r})$**:

$$\mathbf{A}(\bar{r}) = \iiint_V \frac{\mu_0 \mathbf{M}(\bar{r}') \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{4\pi |\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3} \, dv'$$

We can write also write this expression as (trust me!):

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\nabla' \times \mathbf{M}(\bar{r}')}{|\bar{r} - \bar{r}'|} dv' + \frac{\mu_0}{4\pi} \oint\!\!\oint_S \frac{\mathbf{M}(\bar{r}') \times \hat{a}_n}{|\bar{r} - \bar{r}'|} ds'$$

where surface S is the **closed surface** that surrounds material volume V, and unit vector $\hat{a}_n$ is **normal** to this surface.

We find that this is identical to the expression:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} dv' + \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{r}')}{|\bar{r} - \bar{r}'|} ds'$$

if $\mathbf{J}(\bar{r}) = \nabla \times \mathbf{M}(\bar{r})$ and $\mathbf{J}_s(\bar{r}) = \mathbf{M}(\bar{r}) \times \hat{a}_n$.

Therefore, we find that the magnetization of some material, as described by magnetization vector $\mathbf{M}(\bar{r})$, creates **effective** currents $\mathbf{J}_m(\bar{r})$ and $\mathbf{J}_{sm}(\bar{r}_s)$ (where $\bar{r}_s$ indicates points on the material **surface**). We call these effective currents **magnetization currents:**

$$\mathbf{J}_m(\bar{r}) = \nabla \times \mathbf{M}(\bar{r}) \qquad \left[\frac{A}{m^2}\right]$$

$$\mathbf{J}_{sm}(\bar{r}_s) = \mathbf{M}(\bar{r}_s) \times \hat{a}_n \qquad \left[\frac{A}{m}\right]$$

Again, note the **analogy** of these **magnetization** currents with **polarization** charges $\rho_{vp}(\bar{r})$ and $\rho_{sp}(\bar{r})$.

The Magnetic Field

Now that we have defined **magnetization current**, we find that Ampere's Law for fields **within some material** becomes:

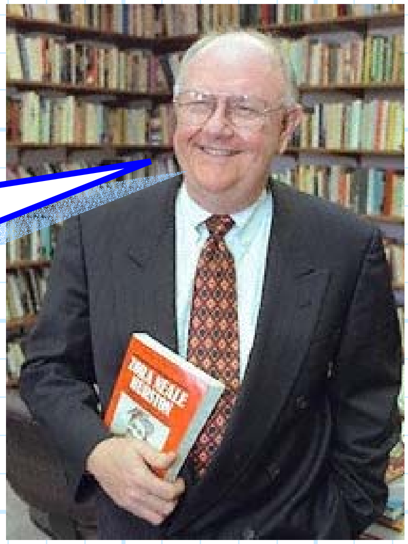
$$\begin{aligned}\nabla \times \mathbf{B}(\bar{r}) &= \mu_0 \left(\mathbf{J}(\bar{r}) + \mathbf{J}_m(\bar{r})\right) \\ &= \mu_0 \left(\mathbf{J}(\bar{r}) + \nabla \times \mathbf{M}(\bar{r})\right)\end{aligned}$$

This of course is **analogous** to the expression we derived for **Gauss's Law** in a dielectric media:

$$\nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_v(\bar{r}) + \rho_{vp}(\bar{r})}{\varepsilon_0} = \frac{\rho_v(\bar{r}) - \nabla \cdot \mathbf{P}(\bar{r})}{\varepsilon_0}$$

Recall that we **removed** the polarization charge from this expression by defining a **new** vector field $\mathbf{D}(\bar{r})$, leaving us with the more **general** expression of Gauss's Law:

$$\nabla \cdot \mathbf{D}(\bar{r}) = \rho_v(\bar{r})$$



Q: *Can we similarly define a* ***new*** *vector field to "take care" of* ***magnetization*** *current ??*

A: Yes! We call this vector field the **magnetic field** $\mathbf{H}(\bar{r})$.

Let's begin by **rewriting** Ampere's Law as:

$$\nabla \times \mathbf{B}(\bar{r}) - \mu_0 \mathbf{J}_m(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

Yuck! Now we see clearly the problem. In **free space**, if we know current distribution $\mathbf{J}(\bar{r})$, we can find the resulting magnetic flux density $\mathbf{B}(\bar{r})$ using the **Biot-Savart** Law:

$$\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathbf{r}}') \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3} dv'$$

But this is the solution for current in **free space**! It is **no longer valid** if some **material** is present!

Q: *Why?*

A: Because, the magnetic flux density produced by current $\mathbf{J}(\bar{r})$ may **magnetize** the material (i.e., produce magnetic dipoles), thus producing **magnetization currents** $\mathbf{J}_m(\bar{r})$.

These magnetization currents $\mathbf{J}_m(\bar{r})$ will **also** produce a magnetic flux density—a **modification** of vector field $\mathbf{B}(\bar{r})$ that is **not** accounted for in the Biot-Savart expression shown above!

To determine the correct solution, we first recall that:

$$\mathbf{J}_m(\bar{r}) = \nabla \times \mathbf{M}(\bar{r})$$

Therefore Ampere's Law is:

$$\nabla \times \mathbf{B}(\bar{r}) - \mu_0 \nabla \times \mathbf{M}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \times \left[\mathbf{B}(\bar{r}) - \mu_0 \mathbf{M}(\bar{r})\right] = \mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \times \left[\frac{\mathbf{B}(\bar{r})}{\mu_0} - \mathbf{M}(\bar{r})\right] = \mathbf{J}(\bar{r})$$

Now let's define a **new** vector field $\mathbf{H}(\bar{r})$, called the **magnetic field**:

$$\mathbf{H}(\bar{r}) \doteq \frac{\mathbf{B}(\bar{r})}{\mu_0} - \mathbf{M}(\bar{r}) \qquad \left[\frac{Amps}{meter}\right]$$

Ampere's Law therefore can be written in terms of the **magnetic field** as:

$$\nabla \times \mathbf{H}(\bar{r}) = \mathbf{J}(\bar{r})$$

Hey! We **know** what the solution to **this** differential equation is! Recall the solution to:

$$\nabla \times \mathbf{B}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

is the **Biot-Savart Law**.

If we make the **substitution**:

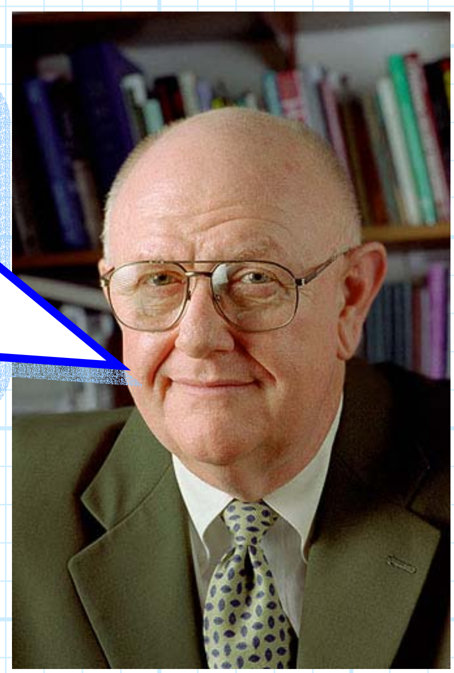
$$\mathbf{H}(\bar{r}) \leftrightarrow \frac{\mathbf{B}(\bar{r})}{\mu_0}$$

we find that both differential **equations** are identical. Therefore their **solutions** are also identical when making the **same** substitution.

Making this substitution into the Biot-Sarvart Law, we find that:

$$\mathbf{H}(\bar{\mathrm{r}}) = \frac{1}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3} \, dv'$$

Q: *Swell. But may I remind you that we were **suppose** to be finding the solution for the &%^@!+*#& **magnetic flux density*** $\mathbf{B}(\bar{r})$!



True! But since we can find $\mathbf{H}(\bar{r})$ from $\mathbf{J}(\bar{r})$, our task **now** is to determine the **relationship** between $\mathbf{B}(\bar{r})$ and $\mathbf{H}(\bar{r})$.

We call the relationship between $\mathbf{B}(\bar{r})$ and $\mathbf{H}(\bar{r})$ a **constitutive equation**. For most media, we find that the magnetization vector $\mathbf{M}(\bar{r})$ is directly **proportional** to the magnetic field $\mathbf{H}(\bar{r})$:

$$\mathbf{M}(\bar{r}) = \chi_m \, \mathbf{H}(\bar{r})$$

where the proportionality coefficient χ_m is the **magnetic susceptibility** of the material.

* Note that for a given magnetic field $\mathbf{H}(\bar{r})$, as χ_m **increases**, the magnetization vector $\mathbf{M}(\bar{r})$ **increases**.

* Magnetic susceptibility χ_m therefore indicates how **susceptible** the material is to **magnetization**.

* In other words, χ_m is a measure of how easily (or difficult) it is to create and align **magnetic dipoles** (from atoms/molecules) within the **material**.

Again, note the **analogy** to electrostatics. We defined earlier **electric** susceptibility χ_e, which indicates how susceptible a material is to **polarization** (i.e., the creation of **electric** dipoles).

We can now determine the relationship between $\mathbf{B}(\bar{r})$ and $\mathbf{H}(\bar{r})$. Using the above expression, we find:

$$\mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu_0} - \mathbf{M}(\bar{r})$$

$$\mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu_0} - \chi_m \mathbf{H}(\bar{r})$$

$$\mathbf{H}(\bar{r}) + \chi_m \mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu_0}$$

$$\mu_0 (1 + \chi_m)\mathbf{H}(\bar{r}) = \mathbf{B}(\bar{r})$$

Hey! Magnetic field $\mathbf{H}(\bar{r})$ and magnetic flux density are related by a **simple constant**!

$$\mathbf{B}(\bar{r}) = \mu \, \mathbf{H}(\bar{r})$$

where:

$\mu \doteq$ material **permeability** $\left[\dfrac{N}{A^2} = \dfrac{Henries}{m}\right]$

$$= \mu_0 (1 + \chi_m)$$

We typically **further** simplify this expression by defining a **relative** permeability:

$$\mu_r \doteq \text{relative permeability}$$
$$= 1 + \chi_m$$

So that:

$$\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r}) = \mu_0 \mu_r \mathbf{H}(\bar{r})$$

In other words, if the **relative** permeability of some material was, say, $\mu_r = 2$, then the **permeability** of the material is **twice** that of the permeability of **free space** (i.e., $\mu = 2\mu_0$). This perhaps is more readily evident when we write:

$$\mu_r = \frac{\mu}{\mu_0}$$

Note that μ and/or μ_r are **proportional** to magnetic susceptibility χ_m. As a result, permeability is likewise an indication of how **susceptible** a material to **magnetization**.

* If $\mu_r = 1$, this susceptibility is that of **free space** (i.e., **none**!).

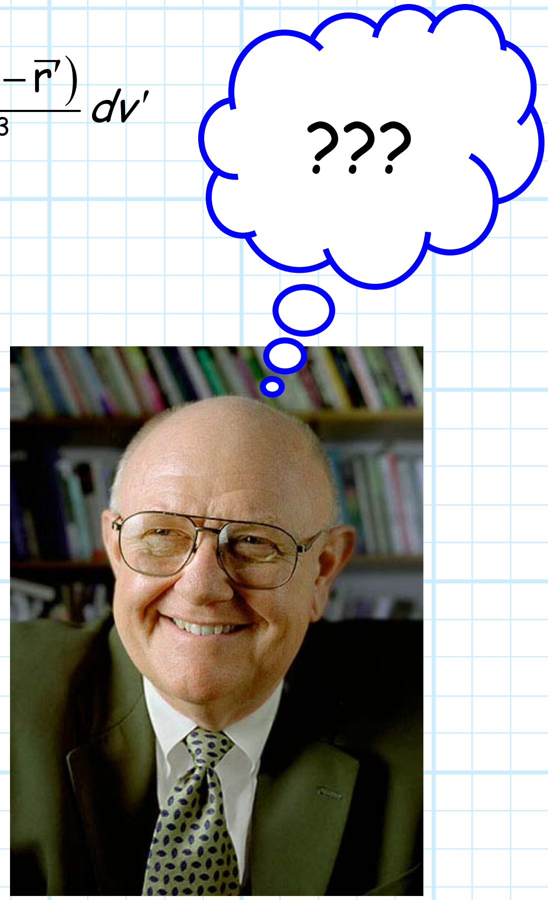
* Alternatively, a **large** μ_r indicates a material that is **easily magnetized**.

* For example, the relative permeability of **iron** is μ_r=4000 !

Now, we are **finally** able to determine the **magnetic flux density** in some **material**, produced by current density $\mathbf{J}(\bar{r})$!

Since $\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r})$ and:

$$\mathbf{H}(\bar{\mathrm{r}}) = \frac{1}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3} dv'$$



we find the desired solution:

$$\mathbf{B}(\bar{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}') \times (\bar{\mathrm{r}} - \bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|^3} dv'$$

Comparing this result with the Biot-Sarvart Law for **free space**, we see that the only difference is that μ_0 has been replaced with μ!

This last result is therefore is a **more general** form of the Biot-Savart Law, giving the correct result for fields within some **material** with permeability μ. Of course, the "material" **could** be free space. However, the expression above will **still** provide the **correct** answer; because for free space $\mu = \mu_0$, thus returning the equation to its **original** (i.e., free space) form!

Summarizing, we can attribute the existence of a **magnetic field** $\mathbf{H}(\bar{r})$ to **conduction** current $\mathbf{J}(\bar{r})$, while we attribute the existence of **magnetic flux density** to the **total** current density, including the magnetization current.

$$\mathbf{J}(\bar{r}) \Rightarrow \mathbf{H}(\bar{r})$$

$$\mathbf{J}(\bar{r}) + \mathbf{J}_m(\bar{r}) \Rightarrow \mathbf{B}(\bar{r})$$

Finally, we again want to note the **analogies** between electrostatics and the magnetostatic expressions derived in this handout:

$$\mathbf{B}(\bar{r}) = \mu_0\mathbf{H}(\bar{r}) + \mu_0\mathbf{M}(\bar{r}) \Leftrightarrow \mathbf{D}(\bar{r}) = \varepsilon_0\mathbf{E}(\bar{r}) + \mathbf{P}(\bar{r})$$

$$\mathbf{B}(\bar{r}) = \mu_0(1+\chi_m)\mathbf{H}(\bar{r}) \Leftrightarrow \mathbf{D}(\bar{r}) = \varepsilon_0(1+\chi_e)\mathbf{E}(\bar{r})$$

$$\mathbf{B}(\bar{r}) = \mu\mathbf{H}(\bar{r}) \Leftrightarrow \mathbf{D}(\bar{r}) = \varepsilon\mathbf{E}(\bar{r})$$

$$\mathbf{B}(\bar{r}) \Leftrightarrow \mathbf{D}(\bar{r})$$

$$\mathbf{H}(\bar{r}) \Leftrightarrow \mathbf{E}(\bar{r})$$

$$\mathbf{M}(\bar{r}) \Leftrightarrow \mathbf{P}(\bar{r})$$

$$\chi_m \Leftrightarrow \chi_e$$

$$\mu \Leftrightarrow \varepsilon$$

Example: Magnetization Currents

Problem:

Consider an **infinite cylinder** made of **magnetic** material. This cylinder is centered along the z-axis, has a **radius of 2 m**, and a **permeability** of $4\mu_0$.

Inside the cylinder there exists a **magnetic flux density**:

$$\mathbf{B}(\bar{r}) = \frac{8\mu_0}{\rho}\,\hat{\mathbf{a}}_\phi \qquad (\rho \leq 1)$$

Determine the **magnetization current** $\mathbf{J}_{sm}(\bar{r}_s)$ flowing **on the surface** of this cylinder, as well as the magnetization current $\mathbf{J}_m(\bar{r})$ flowing **within the volume** of this cylinder.

Solution:

First, we note that we must know the **magnetization vector** $\mathbf{M}(\bar{r})$ in order to find the magnetization currents:

$$\mathbf{J}_m(\bar{r}) = \nabla \times \mathbf{M}(\bar{r}) \qquad \left[\frac{A}{m^2}\right]$$

$$\mathbf{J}_{sm}(\bar{r}_s) = \mathbf{M}(\bar{r}_s) \times \hat{a}_n \qquad \left[\frac{A}{m}\right]$$

But, we must know the **magnetic susceptibility** χ_m and the magnetic field $\mathbf{H}(\bar{r})$ to determine magnetization vector.

$$\mathbf{M}(\bar{r}) = \chi_m \mathbf{H}(\bar{r})$$

Likewise, we need to know the **relative permeability** μ_r to determine magnetic susceptibility:

$$\chi_m = \mu_r - 1$$

and we need to know the **magnetic flux density** $\mathbf{B}(\bar{r})$ to determine the magnetic field:

$$\mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu}$$

But guess what! We **know** the relative permeability μ_r of the material, as well as the magnetic flux density within it!

$$\mu = 4\mu_0, \quad \therefore \mu_r = 4$$

$$\mathbf{B}(\bar{r}) = \frac{8\mu_0}{\rho}\,\hat{\mathbf{a}}_\phi \qquad (\rho \leq 1)$$

Therefore, the **magnetic field** is:

$$\mathbf{H}(\bar{r}) = \frac{\mathbf{B}(\bar{r})}{\mu} = \frac{1}{4\mu_0}\frac{8\mu_0}{\rho}\,\hat{\mathbf{a}}_\phi = \frac{2}{\rho}\,\hat{\mathbf{a}}_\phi$$

and the **magnetic susceptibility** is:

$$\chi_m = \mu_r - 1 = 4 - 1 = 3$$

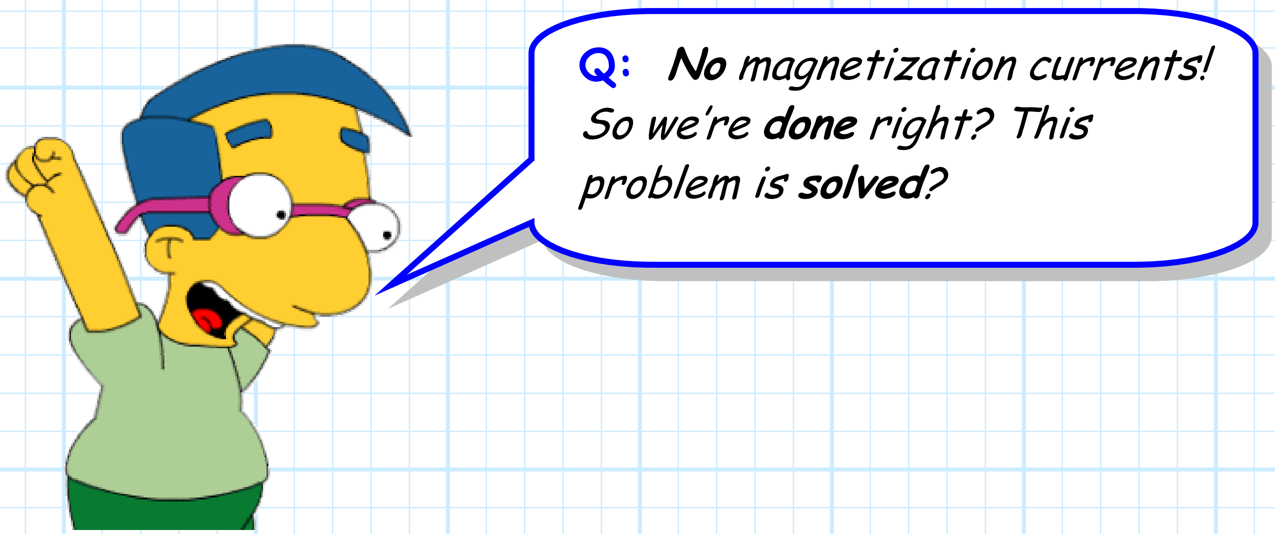
So the **magnetization vector** is:

$$\mathbf{M}(\bar{r}) = \chi_m \mathbf{H}(\bar{r}) = (3)\frac{2}{\rho}\hat{\mathbf{a}}_\phi = \frac{6}{\rho}\hat{\mathbf{a}}_\phi$$

Now (**finally!**) we can determine the **magnetization currents**:

$$\begin{aligned}\mathbf{J}_m(\bar{r}) &= \nabla \times \mathbf{M}(\bar{r}) \\ &= \nabla \times \left(\frac{6}{\rho}\hat{\mathbf{a}}_\phi\right) \\ &= 0\end{aligned}$$

The volume magnetization current density is **zero**—there is no magnetization current flowing **within** the cylinder!



A: Not hardly! Although there are no magnetization currents flowing **within** the cylinder, there might be magnetization currents flowing on the cylinder **surface** (i.e., $\mathbf{J}_{sm}(\bar{r}_s)$) !

$$\mathbf{J}_{sm}(\bar{r}_s) = \mathbf{M}(\bar{r}_s) \times \hat{\mathbf{a}}_n$$

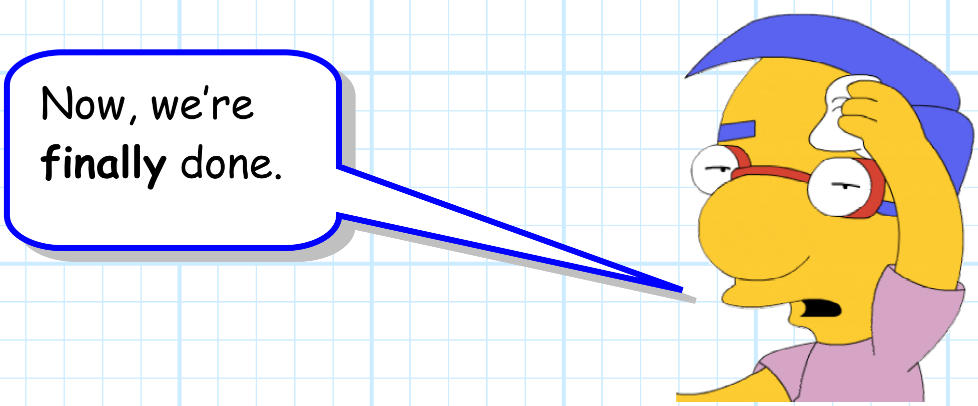
Note for this problem, the unit vector normal to the surface of the cylinder is $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_\rho$.

Likewise, the magnetization vector **evaluated at the cylinder surface** (i.e., at $\rho = 2$) is:

$$\mathbf{M}(\bar{r}_s) = \mathbf{M}(\rho = 2) = \left.\frac{6}{\rho}\,\hat{\mathbf{a}}_\phi\right|_{\rho=2} = 3\,\hat{\mathbf{a}}_\phi$$

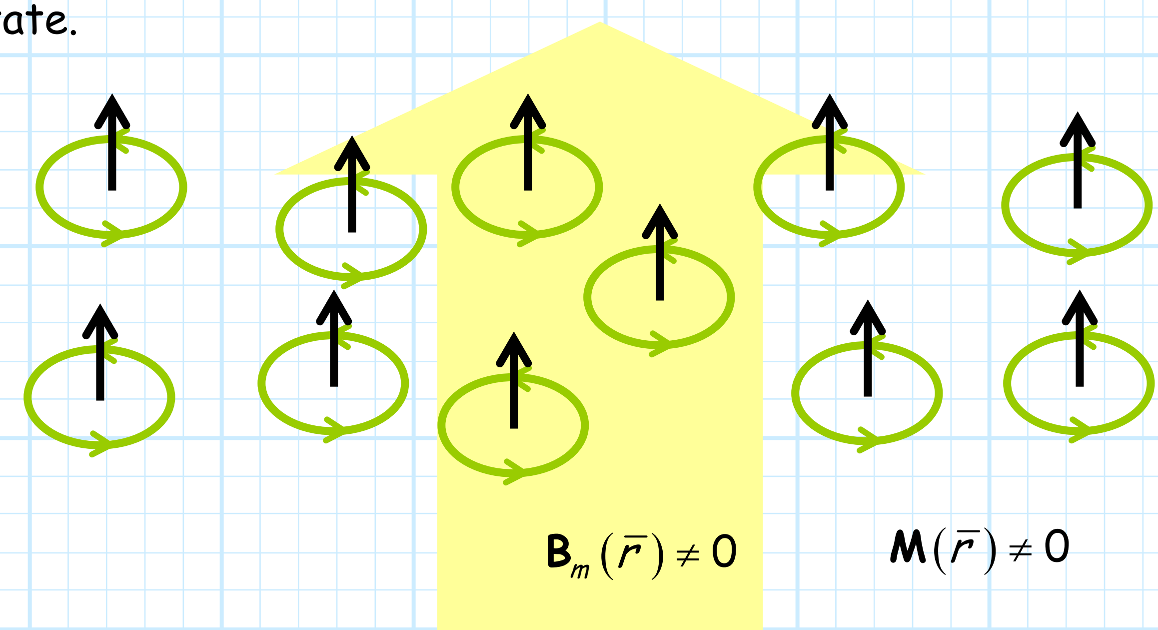
Therefore, the **magnetization current density** on the cylinder surface is:

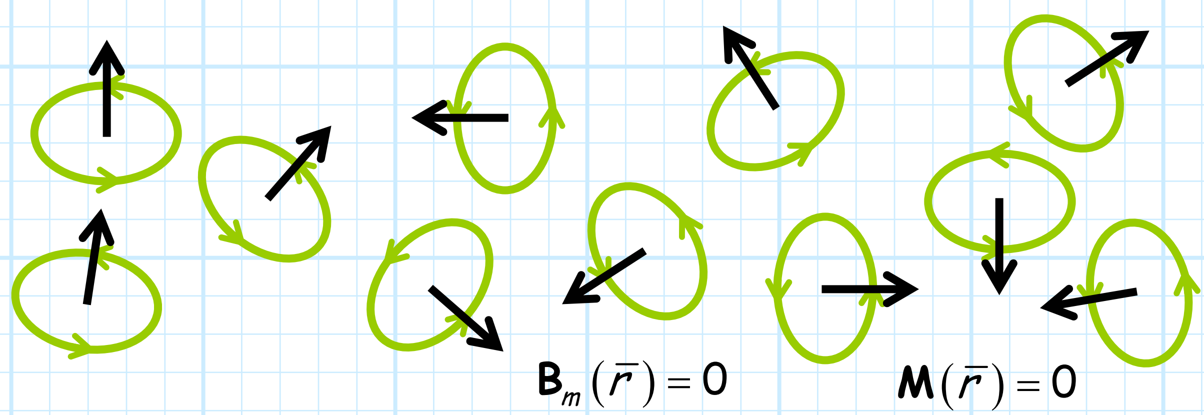
$$\begin{aligned}\mathbf{J}_{sm}(\rho = 2) &= \mathbf{M}(\rho = 2) \times \hat{\mathbf{a}}_n \\ &= 3\,\hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_\rho \\ &= -3\,\hat{\mathbf{a}}_z \qquad [A/m]\end{aligned}$$



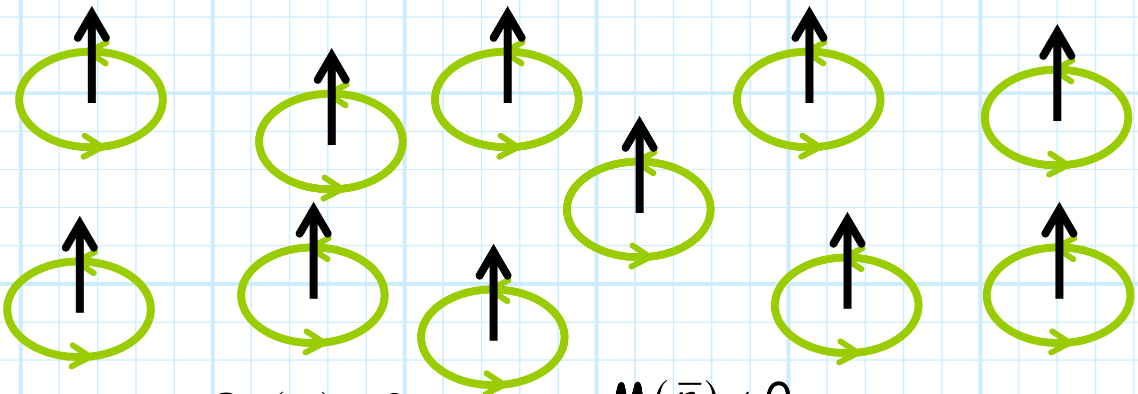
Permanent Magnets

For **most** magnetic material (i.e., where $\mu \neq \mu_0$), we find that the magnetization vector $\mathbf{M}(\bar{r})$ will return to **zero** when a magnetization field $\mathbf{B}_m(\bar{r})$ is removed. In other words, the **magnetic dipoles** will vanish, or at least return to their random state.

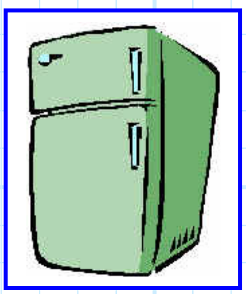




However, some magnetic material, called **ferromagnetic** material, will **retain** its dipole orientation, even when the magnetizing field is removed !



$\mathbf{B}_m(\bar{r}) = 0$ $\mathbf{M}(\bar{r}) \neq 0$



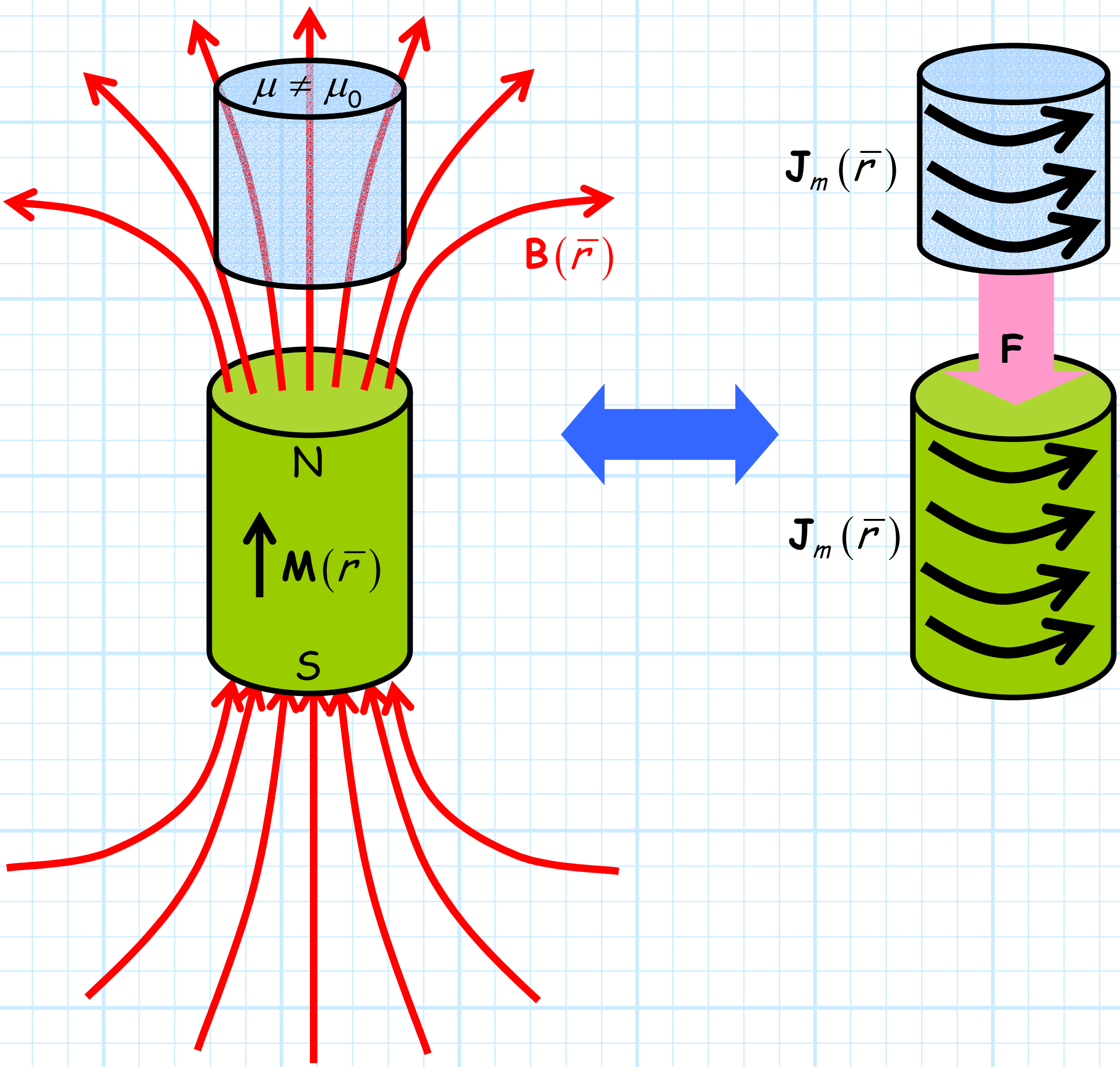
In this case, a **permanent magnet** is formed (just like the ones you stick on your fridge)!

Ferromagnetic materials have **numerous applications**. For example, they will **attract** magnetic material.

Q: *How?*

A: A permanent magnet will of course produce **everywhere** a magnetic flux density $\mathbf{B}(\bar{r})$, which we can **either** attribute to the magnetic **dipoles** with in the material, **or** to the equivalent magnetic **current** $\mathbf{J}_m(\bar{r})$.

The magnetic flux density produced by the magnet will act as a **magnetizing** field for some **other** magnetic material nearby, thus creating a **second** magnetization current $\mathbf{J}_m(\bar{r})$ within the nearby material. The magnetization currents of the material and the magnet will **attract**!

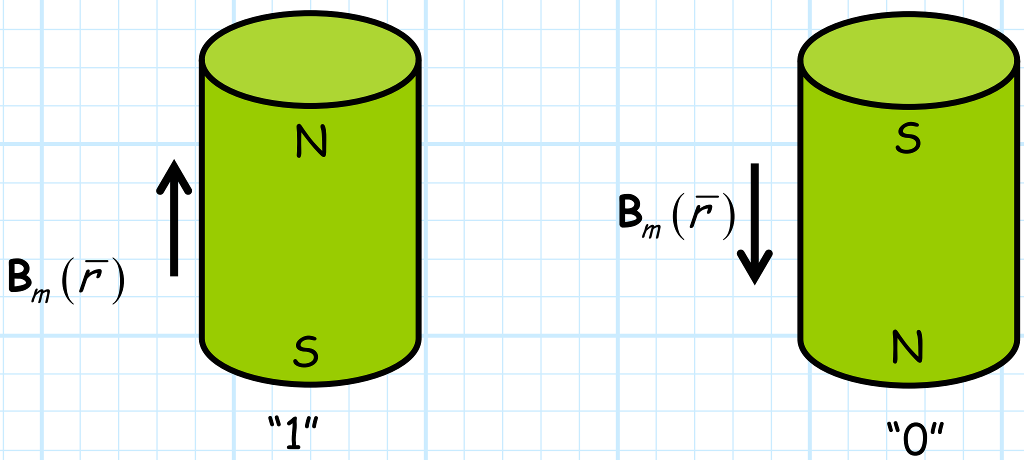


Another interesting application of ferromagnetic material is in non-volatile **data storage** (e.g., tape or disk). Ferromagnetics can be used as **binary memory** !

Q: *How?*

A: Recall that the magnetization vector in ferromagnetic material retains its direction after the magnetizing field $\mathbf{B}_m(\bar{r})$ has been removed. In other words, it "**remembers**" the direction of the magnetizing field.

We can assign each of **two** different magnetizing directions, therefore, a **binary** state:



If ferromagnetic material is **embedded** in a tape or disk, we can magnetize (e.g., **write**) small sections of the media, or detect the magnetization (e.g., **read**) small sections of the media.

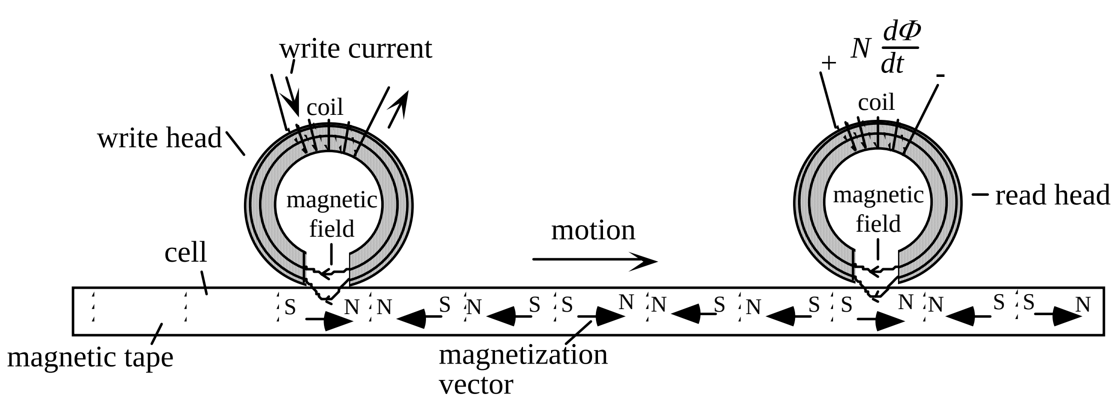


Figure 8-21

Field Equations in Magnetic Materials

Now that we have defined a magnetic field $\mathbf{H}(\bar{r})$ and material permeability $\mu(\bar{r})$, we can write the magnetostatic (point form) equations for fields in **magnetic material**.

$$\nabla \mathrm{x} \mathbf{H}(\bar{r}) = \mathbf{J}(\bar{r})$$

$$\nabla \cdot \mathbf{B}(\bar{r}) = 0$$

$$\mathbf{B}(\bar{r}) = \mu(\bar{r})\mathbf{H}(\bar{r})$$

We likewise can express these equations in **integral form** as:

$$\oint_C \mathbf{H}(\bar{r}) \cdot \overline{d\ell} = I_{enc}$$

$$\oiint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} = 0$$

$$\mathbf{B}(\bar{r}) = \mu(\bar{r})\mathbf{H}(\bar{r})$$

First, note the **new** form of **Ampere's Law**:

$$\oint_C \mathbf{H}(\bar{r}) \cdot \overline{d\ell} = I_{enc}$$

Where I_{enc} is the conduction current only (i.e., it does not include magnetization current!).

Again, note the **analogies** to the new form of **Gauss's Law** we derived for electrostatics:

$$\iint_S \mathbf{D}(\bar{r}) \cdot \overline{ds} = Q_{enc}$$

where Q_{enc} is the free-charge enclosed by surface S.

Perhaps the most important result of expressing magnetostatic fields in terms of material **permeability** $\mu(\bar{r})$ is that we **do not** have to **rederive** any of the results from Chapter 7!

In Chapter 7, the "material" we were concerned with was **free space**. The permeability of free space is by definition, $\mu(\bar{r}) = \mu_0$.

If the material is **not** free space, then we simply **change** the results of Chapter 7 to reflect the **correct value** of **permeability** $\mu(\bar{r})$.

For example, we found that the Biot-Savart Law becomes,:

$$\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu I}{4\pi} \oint_C \frac{\overline{d\ell'} \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3}$$

magnetic vector potential is,:

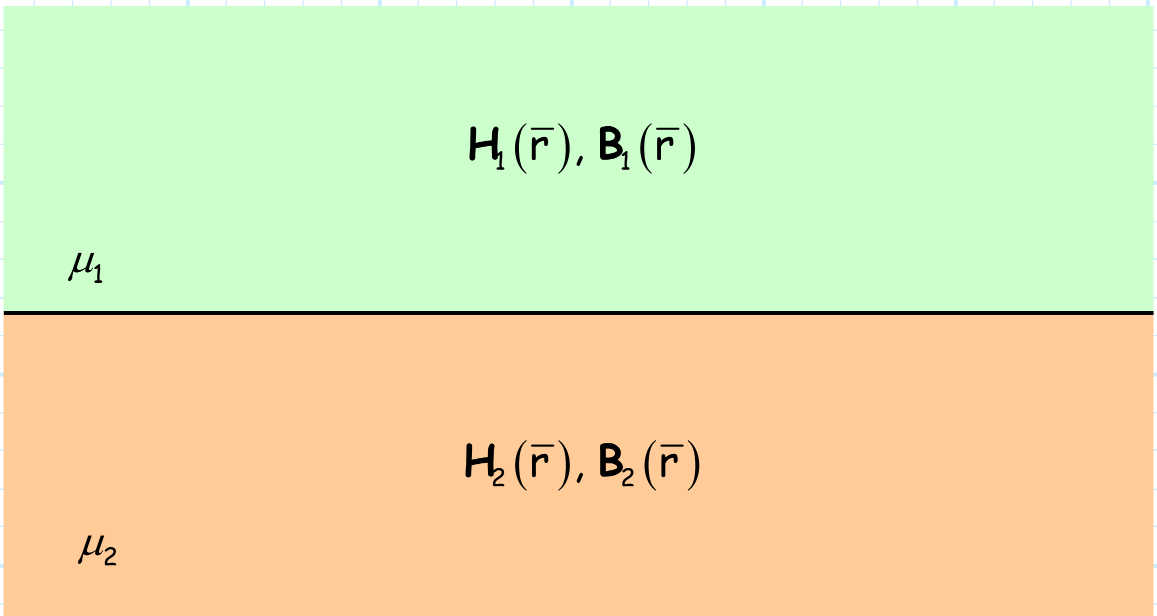
$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} \, dv'$$

or the magnetic flux produced by a infinite line current is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu I}{2\pi \rho} \hat{a}_\phi$$

Magnetic Boundary Conditions

Consider the **interface** between two different materials with dissimilar permeabilities:

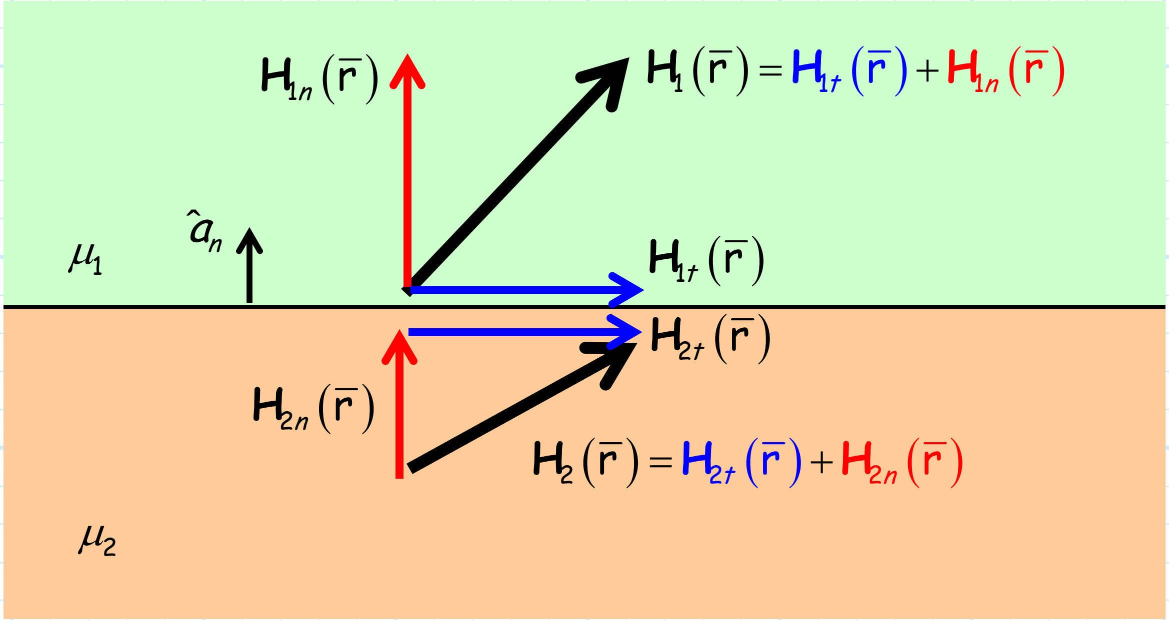


Say that a magnetic field and a magnetic flux density is present in **both** regions.

Q: *How are the fields in dielectric **region 1** (i.e., $\mathbf{H}_1(\bar{r}), \mathbf{B}_1(\bar{r})$) related to the fields in **region 2** (i.e., $\mathbf{H}_2(\bar{r}), \mathbf{B}_2(\bar{r})$)?*

A: They must satisfy the **magnetic boundary conditions** !

First, let's write the fields **at the interface** in terms of their **normal** (e.g., $\mathbf{H}_n(\bar{r})$) and **tangential** (e.g., $\mathbf{H}_t(\bar{r})$) vector components:



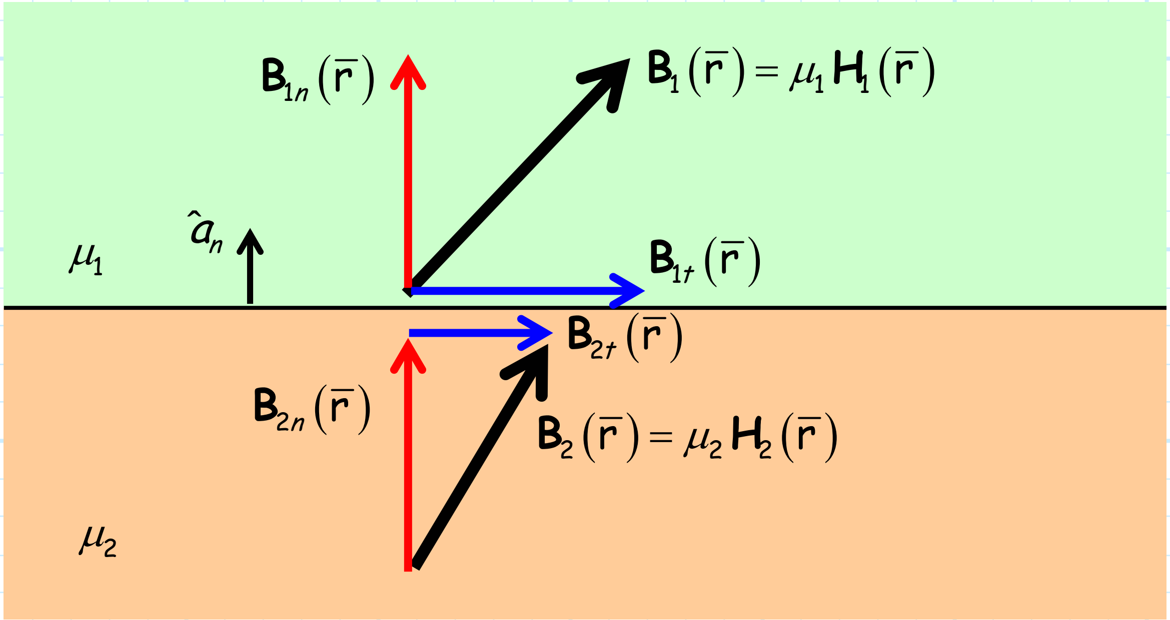
Our first boundary condition states that the **tangential** component of the magnetic field is **continuous** across a boundary. In other words:

$$\mathbf{H}_{1t}(\bar{r}_b) = \mathbf{H}_{2t}(\bar{r}_b)$$

where $\bar{r}_b$ denotes to **any** point along the interface (e.g., material boundary).

The **tangential** component of the magnetic field on **one** side of the material boundary is **equal** to the tangential component on the **other** side !

We can likewise consider the **magnetic flux densities** on the material interface in terms of their **normal** and **tangential** components:



The second magnetic boundary condition states that the **normal** vector component of the **magnetic flux density** is **continuous** across the material boundary. In other words:

$$\mathbf{B}_{1n}(\bar{r}_b) = \mathbf{B}_{2n}(\bar{r}_b)$$

where $\bar{r}_b$ denotes **any** point along the interface (i.e., the material boundary).

Since $\mathbf{B}(\bar{r}) = \mu\,\mathbf{H}(\bar{r})$, these boundary conditions can **likewise** be expressed as:

$$\mathbf{H}_{1t}(\bar{r}_b) = \mathbf{H}_{2t}(\bar{r}_b)$$

$$\frac{\mathbf{B}_{1t}(\bar{r}_b)}{\mu_1} = \frac{\mathbf{B}_{2t}(\bar{r}_b)}{\mu_2}$$

and as:

$$\mathbf{B}_{1n}(\bar{r}_b) = \mathbf{B}_{2n}(\bar{r}_b)$$

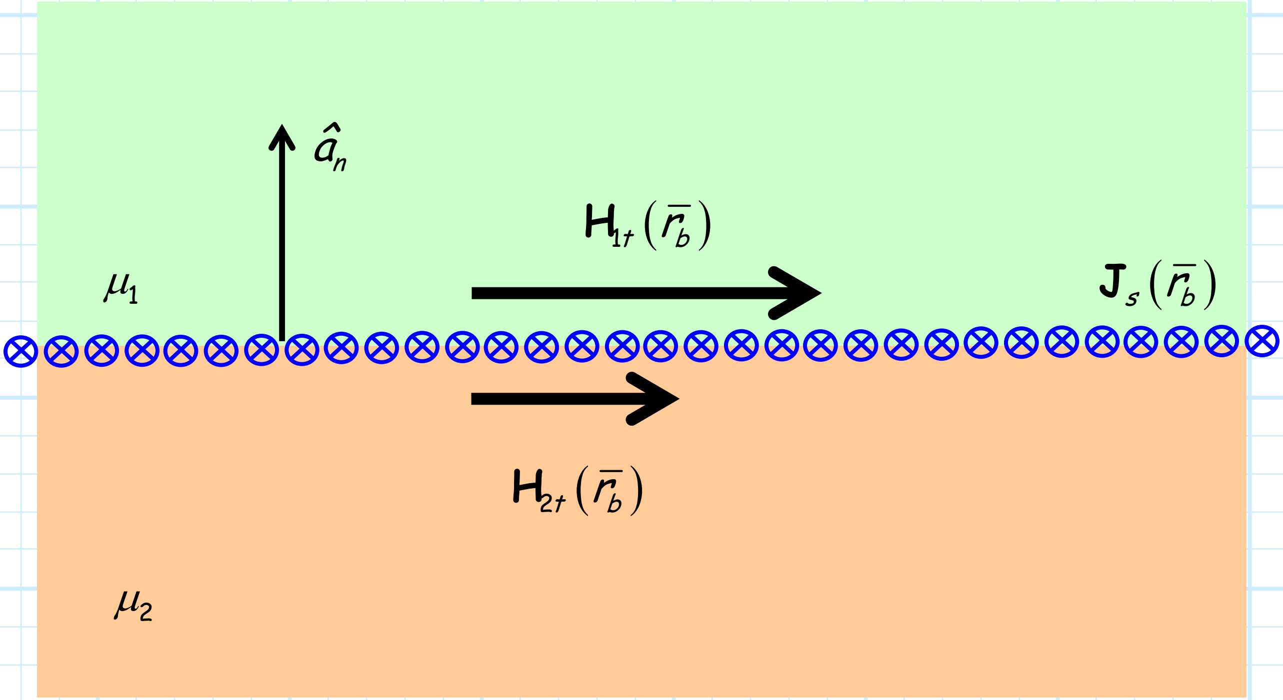
$$\mu_1\mathbf{H}_{1n}(\bar{r}_b) = \mu_2\mathbf{H}_{2n}(\bar{r}_b)$$

Note again the perfect **analogy** to the boundary conditions of **electrostatics**!

Finally, recall that if a layer of **free charge** were lying at a dielectric boundary, the boundary condition for electric flux density was **modified** such that:

$$\hat{a}_n \cdot \left[\mathbf{D}_1(\bar{r}_b) - \mathbf{D}_2(\bar{r}_b) \right] = \rho_s(\bar{r}_b)$$
$$D_{1n}(\bar{r}_b) - D_{2n}(\bar{r}_b) = \rho_s(\bar{r}_b)$$

There is an **analogous** problem in magnetostatics, wherein a **surface current** is flowing at the interface of two magnetic materials:



In this case the tangential components of the magnetic field will **not** be continuous!

Instead, they are related by the boundary condition:

$$\hat{a}_n \times \left(\mathbf{H}_1(\bar{r}_b) - \mathbf{H}_2(\bar{r}_b)\right) = \mathbf{J}_s(\bar{r}_b)$$

This expression means that:

1) $\mathbf{H}_{1t}(\bar{r}_b)$ and $\mathbf{H}_{2t}(\bar{r}_b)$ point in the **same** direction.

2) $\mathbf{H}_{1t}(\bar{r}_b)$ and $\mathbf{H}_{2t}(\bar{r}_b)$ are **orthogonal** to $\mathbf{J}_s(\bar{r}_b)$.

3) The difference between $\left|\mathbf{H}_{1t}(\bar{r}_b)\right|$ and $\left|\mathbf{H}_{2t}(\bar{r}_b)\right|$ is $\left|\mathbf{J}_s(\bar{r}_b)\right|$.

Recall that $\mathbf{H}(\bar{r})$ and $\mathbf{J}_s(\bar{r})$ have the same units—**Amperes/meter**!

Note for this case, the boundary condition for the magnetic flux density remains **unchanged**, i.e.:

$$\mathbf{B}_{1n}(\bar{r}_b) = \mathbf{B}_{2n}(\bar{r}_b)$$

regardless of $\mathbf{J}_s(\bar{r}_b)$.